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Multi-parametric deformed Heisenberg algebras: a route to complexity

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Abstract

We introduce a generalization of the Heisenberg algebra which is written in terms of a functional of one generator of the algebra, $f(J_0)$, that can be any analytical function. When f is linear with slope θ, we show that the algebra in this case corresponds to q-oscillators for $q^2 = \tan\theta$. The case where f is a polynomial of order n in J_0 corresponds to an n-parameter deformed Heisenberg algebra. The representations of the algebra, when f is any analytical function, are shown to be obtained through the study of the stability of the fixed points of f and their composed functions. The case when f is a quadratic polynomial in J_0, the simplest nonlinear scheme which is able to create chaotic behaviour, is analysed in detail and special regions in the parameter space give representations that cannot be continuously deformed to representations of Heisenberg algebra.

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1. Introduction

Quantum algebras first appeared in the algebraic Bethe ansatz approach to quantum integrable one-dimensional models [1]. Since then, there have been several attempts to apply them in a broad range of physical phenomena [2].

Associated with the omnipresent harmonic oscillator there is an algebra known as the Heisenberg algebra. The simple structure of this algebra, that is described in terms of creation and annihilation operators, and its particle interpretation has promoted it to a paradigmatic tool in the second quantization approach.

A connection between these two topics appeared soon after the discovery of quantum algebras, when it was found out that a generalization of the Heisenberg algebra, known as q-oscillators, was necessary in order to realize $su_q(2)$ through the Jordan–Schwinger method [3].

Guided, in part, by the wide range of physical applicability of the Heisenberg algebra there have been efforts in the last 10 years to analyse possible physical relevance of q-oscillators or deformed Heisenberg algebras [4]. The expected physical properties of toy systems described by these generalized Heisenberg algebras were analysed and indications on how to solve an old puzzle in physics were obtained [5].

Recently, an algebra, called logistic algebra, that is a generalization of the Heisenberg algebra where the eigenvalues of one generator of the algebra (the one that generalizes the number operator) are given by functional iterations of the logistic function, was introduced. This algebra has finite-and infinite-dimensional representations associated with the cycles of the logistic map and infinite-dimensional representations related to the chaotic band [6, 7].

A quantum solid Hamiltonian whose collective modes of vibration are described by oscillators satisfying the logistic algebra was constructed and the thermodynamic properties of this model in the two-cycle and in a specific chaotic region of the logistic map were analysed. It is interesting to mention that in the chaotic band this model shows a curious hybrid behaviour mixing classical and quantum behaviour, showing how a quantum system can present a non-standard quantum behaviour [7].

In this paper, a generalization of the logistic algebra is constructed in such a way that the eigenvalues of one generator are given by a functional iteration of a starting number. This functional could be any analytical function, but, in order to study the properties of this algebra in detail, this function is taken as a polynomial of order n.

When the functional, $f(J_0)$, is linear in J_0, where J_0 is the Hermitian generator of the algebra, i.e. $f(J_0) = r\,J_0 + s$, $r = q^2$ is shown to correspond to the q-deformed Heisenberg algebra or q-oscillators. The general case, $f(J_0) = \sum_{i=0}^{n} r_i\,J_0^i$, is an n-parameter deformed Heisenberg algebra. This algebra is, therefore, a multi-parametric deformation of the Heisenberg algebra.

The representation theory is presented in detail for the linear and quadratic cases since they are the paradigmatic ones. It is shown that the essential tool in order to find the representations of the algebra is the analysis of the stability of the fixed points of the polynomial f and their composed functions.

Related to the cycles of period $1, 2, 4, \ldots$ there are finite- and infinite-dimensional representations of the algebra. The weights of the finite-dimensional representations are given exactly by the lowest values of the cycles.

In the next section we present the general algebra and the general representation theory. In section 3 we analyse the linear case, its representations and its connection to q-oscillators. The nonlinear case or two-parameter deformed Heisenberg algebra is presented in section 4, where the essential role played by the analysis of the stability of the fixed points of the polynomial f and their composed functions in order to obtain the finite- and infinite-dimensional representations of the algebra becomes evident. In section 5 we present our final comments and also introduce a generalization of $su(2)$ in the sense discussed in this paper.

2. Generalized Heisenberg algebra

Let us consider an algebra generated by J_0, $J_\pm$ described by the relations

$$J_0\,J_+ = J_+\,f(J_0) \tag{1}$$

$$J_-\,J_0 = f(J_0)\,J_- \tag{2}$$

$$\left[J_+, J_-\right] = J_0 - f(J_0). \tag{3}$$

By hypothesis, $J_- = J_+^\dagger$ and $J_0^\dagger = J_0$, and $f(J_0)$ is a general analytic function of J_0. The case where $f(J_0) = r\,J_0\,(1 - J_0)$ was analysed in [6] and [7]. The above algebra relations are constructed in order that the eigenvalues of operator J_0 are given by an iteration of an initial value as will be clear in a moment.

Let us now show that the operator

$$C = J_+ J_- - J_0 = J_- J_+ - f(J_0) \tag{4}$$

is a Casimir operator of the algebra. Using the algebraic relations in equations (1)–(3) it is easy to see that

$$[C, J_0] = \left[C, J_\pm\right] = 0 \tag{5}$$

i.e. C is one Casimir operator of the algebra.

We start now analysing the representation theory of the algebra when the function $f(J_0)$ is a general analytic function of J_0. In this section we obtain the general equations for an n-dimensional representation and in the next sections we solve these equations for linear and quadratic polynomials $f(J_0)$, finding the finite- and infinite-dimensional representations for the linear and quadratic cases that are the paradigmatic ones.

We assume we have an n-dimensional irreducible representation of the algebra given in equations (1)–(3). The Hermitian operator J_0 can be diagonalized. Consider the state $|0\rangle$ with the lowest eigenvalue of J_0

$$J_0 \, |0\rangle = \alpha_0 \, |0\rangle. \tag{6}$$

For each value of α_0 and the parameters of the algebra we have a different vacuum that for simplicity will be denoted by $|0\rangle$. Moreover, it will be clear in the next sections, when we shall solve the representation theory for the linear and quadratic polynomials $f(J_0)$, that the allowed values of α_0 depend on the parameters of the algebra.

Let $|m\rangle$ be a normalized eigenstate of J_0,

$$J_0|m\rangle = \alpha_m|m\rangle. \tag{7}$$

Applying equation (1) to $|m\rangle$ we have

$$J_0(J_+|m\rangle) = J_+ f(J_0)|m\rangle = f(\alpha_m)(J_+|m\rangle). \tag{8}$$

Thus, we see that $J_+|m\rangle$ is a J_0 eigenvector with eigenvalue $f(\alpha_m)$. Starting from $|0\rangle$ and applying J_+ successively to $|0\rangle$ we create different states with the J_0 eigenvalue given by

$$J_0 \left(J_+^m |0\rangle\right) = f^m(\alpha_0) \left(J_+^m |0\rangle\right) \tag{9}$$

where $f^m(\alpha_0)$ denotes the mth iterate of f. Since the application of J_+ creates a new vector, whose respective J_0 eigenvalue has iterations of α_0 through f increased by one unit, it is convenient to define the new vectors $J_+^m|0\rangle$ as proportional to $|m\rangle$ and we then call J_+ a raising operator. Note that

$$\alpha_m = f^m(\alpha_0) = f(\alpha_{m-1}) \tag{10}$$

where m denotes the number of iterations of α_0 through f.

Following the same procedure for J_-, applying equation (2) to $|m+1\rangle$, we have

$$J_- J_0|m+1\rangle = f(J_0)\,(J_-|m+1\rangle) = \alpha_{m+1}\,(J_-|m+1\rangle)\,. \tag{11}$$

This shows that $J_-|m+1\rangle$ is also a J_0 eigenvector with eigenvalue α_m. Then, $J_-|m+1\rangle$ is proportional to $|m\rangle$, showing that J_- is a lowering operator.

Since we consider α_0 the lowest J_0 eigenvalue, we require

$$J_- \, |0\rangle = 0. \tag{12}$$

As was shown in [7], depending on the function f and its initial value α_0, it may happen that the J_0 eigenvalue of state $|m+1\rangle$ is lower than that of state $|m\rangle$. Thus, as we exemplify in section 4 of this paper, given an arbitrary analytical function f (and its associated algebra in equations (1)–(3)) in order to satisfy equation (12), the allowed values of α_0 are chosen in such a way that the iterations $f^m(\alpha_0)$ ($m \geqslant 1$) are always bigger than α_0. Then, equation (12) must be checked for every function f, giving consistent vacua for specific values of α_0. This analysis is made in sections 3 and 4 where we find the parameter regions with consistent representations.

In general we obtain

$$J_0\,|m-1\rangle = f^{m-1}(\alpha_0)\,|m-1\rangle \qquad m = 1, 2, \ldots \tag{13}$$

$$J_+\,|m-1\rangle = N_{m-1}\,|m\rangle \tag{14}$$

$$J_-\,|m\rangle = N_{m-1}\,|m-1\rangle \tag{15}$$

where $N_{m-1}^2 = f^m(\alpha_0) - \alpha_0$. We observe that if we put $m = 0$ in equation (15) then N_{-1} is equal to zero, which is consistent with equation (12). Equations (13)–(15) are easily proven by induction. In order to verify equations (13)–(15) for $m = 1$, apply equation (1) to the state vector $|0\rangle$ obtaining $J_0\,(J_+|0\rangle) = f(\alpha_0)\,(J_+|0\rangle)$. Thus, we define $|1\rangle \equiv \frac{1}{N_0} J_+|0\rangle$ where N_0 is a constant to be determined. It is easy to see that $J_0|1\rangle = f(\alpha_0)|1\rangle$. The constant N_0 can be determined by imposing that the state vector $|1\rangle$ has unit norm and, with the use of equation (3), we obtain $N_0^2 = f(\alpha_0) - \alpha_0$. As the last step of this check apply equation (3) to the state $|0\rangle$. Using equations (6) and (12) we obtain $J_-|1\rangle = N_0|0\rangle$. Then, equations (13)–(15) are verified for $m = 1$.

Now, suppose equations (13)–(15) are valid for m. Apply J_0 to equation (14) and use equation (1) on the left-hand side; this gives

$$J_0\,|m\rangle = f^m(\alpha_0)\,|m\rangle. \tag{16}$$

Applying equation (1) to the state $|m\rangle$ and using equation (16) we are allowed to suppose that there exists a state vector $|m+1\rangle$ such that

$$|m+1\rangle = \frac{1}{C(m)}\,J_+|m\rangle \tag{17}$$

where $C(m)$ is a constant. This constant is determined by imposing that the state vector $|m+1\rangle$ has unit norm

$$\begin{aligned} 1 = \langle m+1|m+1\rangle &= \frac{1}{C(m)^2}\,\langle m|J_-\,J_+|m\rangle \\ &= \frac{1}{C(m)^2}\left[\langle m|J_+\,J_-|m\rangle + \langle m|(-J_0 + f(J_0))|m\rangle\right] \\ &= \frac{1}{C(m)^2}\left(N_{m-1}^2 - f^m(\alpha_0) + f^{m+1}(\alpha_0)\right) \end{aligned} \tag{18}$$

which gives $C(m)^2 = N_m^2 = f^{m+1}(\alpha_0) - \alpha_0$.

Applying equation (2) to $|m\rangle$ and using equations (13)–(15) and the value of N_m we obtain the last equation we wanted. Putting everything together we recover equations (13)–(15) for $m \mapsto m+1$ and the proof is complete.

Note that equations (13)–(15) define a general n-dimensional representation for the algebra in equations (1)–(3). In order to solve it, i.e. to construct the conditions under which we have finite- and infinite-dimensional representations, we have to specify the functional $f(J_0)$. It is easy to see that if we choose $f(J_0) = J_0 + 1$ the algebra given by equations (1)–(3) becomes with this choice the Heisenberg algebra for A, $A^\dagger$ and $N = A^\dagger A$ where $A = J_-$, $A^\dagger = J_+$ and $N = J_0$. Note that the Casimir operator in equation (4), that in the general case has eigenvalue equal to $-\alpha_0$, becomes in this case $C = A^\dagger A - N$, which is identically null. We shall see in the next section that the choice $f(J_0) = r\,J_0 + s$ corresponds to a one-parameter deformed Heisenberg algebra and if we take a functional with linear and quadratic terms (besides a constant term) we have a quadratic Heisenberg algebra or a two-parameter deformed Heisenberg algebra that will be analysed in section 4.

Another very interesting observation is that, as mentioned at the beginning of this section, the algebraic relations equations (1) and (2) are constructed in such a way that the eigenvalues of operator J_0 are iterations of an initial value α_0 through the function f as shown in equation (13). Then, the increasing complexity of function f will correspond to an increasing complex behaviour of the eigenvalues of J_0 [8]. In fact, as already shown in [6,7], choosing the logistic map for f, it could give rise to a chaotic behaviour of the eigenvalue of J_0. Moreover, as will be clear in the next sections, it is this iteration aspect of the algebra that will allow us to find the representations through the analysis of the stability of the fixed points of the function f and their composed functions.

3. The linear case

In this section we are going to find the representations for the algebra defined by the relations given in equations (1)–(3) considering $f(J_0) = r\,J_0 + s$. The algebra relations can be rewritten for this case as

$$[J_0, J_+]_r = s\,J_+ \tag{19}$$

$$\left[J_0, J_-\right]_{r^{-1}} = -\frac{s}{r} J_- \tag{20}$$

$$\left[J_+, J_-\right] = (1-r)\,J_0 - s \tag{21}$$

where $[a, b]_r \equiv a\,b - r\,b\,a$ is the r-deformed commutation of two operators a and b.

It is very simple to realize that, for $r = 1$ and s arbitrary, the above algebra is the Heisenberg algebra for A, $A^\dagger$ and N where $A = J_-/\sqrt{s}$, $A^\dagger = J_+/\sqrt{s}$ and $N = J_0/s$. In this case the Casimir operator given in equation (4) is null. Then, for general r and s the algebra defined in equations (19)–(21) is a one-parameter deformed Heisenberg algebra and generally speaking the algebra given in equations (1)–(3) is a generalization of the Heisenberg algebra.

It is easy to see for the general linear case that

$$\begin{aligned} f^m(\alpha_0) &= r^m\,\alpha_0 + s\,(r^{m-1} + r^{m-2} + \cdots + 1) \\ &= r^m \alpha_0 + s\,\frac{r^m - 1}{r-1} \end{aligned} \tag{22}$$

thus

$$N_{m-1}^2 = f^m(\alpha_0) - \alpha_0 = [m]_r\; N_0^2 \tag{23}$$

where $[m]_r \equiv (r^m - 1)/(r-1)$ is the Gauss number of m and $N_0^2 = \alpha_0\,(r-1) + s$.

Let us search for finite-dimensional representations of the linear Heisenberg algebra. Our approach is the following: we start from the vacuum state $|0\rangle$ and apply repeatedly the operator J_+ arriving, for specific values of α_0, r and s, eventually at $J_+|n-1\rangle = 0$ for an n-dimensional representation. From equation (14) we see that the set of parameters providing an m-dimensional representation, using equation (23), is computed from

$$\begin{aligned} &N_0^2 = \alpha_0\,(r-1) + s \;>\; 0 \\ &N_1^2 = [2]_r\; N_0^2 \;>\; 0 \\ &\cdots \\ &N_{m-2}^2 = [m-1]_r\; N_0^2 \;>\; 0 \\ &N_{m-1}^2 = [m]_r\; N_0^2 = 0. \end{aligned} \tag{24}$$

The solutions for $[m]_r = 0$ are given by $r = \exp(2\pi \mathrm{i}k/m)$ for $k = 1, 2, \ldots, m-1$ ($k = 0$ corresponds to the Heisenberg algebra that we are not considering at the moment) but since J_0

is taken to be Hermitian, the only interesting finite-dimensional solution is a two-dimensional ($m = 2$) representation with $r = -1$ and $s > 2\alpha_0$. There is of course a trivial one-dimensional representation where the weight of the representation is the fixed point $\alpha_0 = \alpha^* = s/(1-r)$ and $r \in (-1, 1) \cup (1, \infty)$. We have also a marginal uninteresting one-dimensional solution obtained for $r \to \infty$ and $s/r^2 =$ finite.

The infinite-dimensional solutions are more interesting. In this case we must solve the following set of equations:

$$N_m^2 > 0 \qquad \forall m \qquad m = 0, 1, 2, \ldots. \tag{25}$$

Apart from the Heisenberg algebra given by $r = 1$, the solutions are

$$\text{type I}: \quad r > 1 \qquad \text{and} \qquad \alpha_0 > \frac{s}{1-r} \tag{26a}$$

or

$$\text{type II}: \quad -1 < r < 1 \qquad \text{and} \qquad \alpha_0 < \frac{s}{1-r} \tag{26b}$$

with matrix representations

$$J_0 = \begin{pmatrix} \alpha_0 & 0 & 0 & 0 & \ldots \\ 0 & \alpha_1 & 0 & 0 & \ldots \\ 0 & 0 & \alpha_2 & 0 & \ldots \\ 0 & 0 & 0 & \alpha_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ N_0 & 0 & 0 & 0 & \ldots \\ 0 & N_1 & 0 & 0 & \ldots \\ 0 & 0 & N_2^{\o} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad J_- = J_+^\dagger. \tag{27}$$

Note that for type I solutions the eigenvalues of J_0, as can be easily computed from equations (13) and (10), go to infinity as we consider eigenvectors $|m\rangle$ with increasing value of m. Instead, for type II solutions the eigenvalues go to the value $s/(1-r)$, the fixed point of f, as the state $|m\rangle$ increase.

The reason for this asymptotic behaviour of the eigenvalues of J_0 is simple. It is clear from equations (13) and (10) that the eigenvalues of J_0 are given by the functional iteration of $f(\alpha) = r\,\alpha + s$ for the starting number α_0. Moreover, the stability of the fixed point of $f(\alpha)$ is directly related to the asymptotic behaviour of the eigenvalue of J_0. If the fixed point of $f(\alpha)$ is stable ($-1 < r < 1$) or unstable ($r > 1$) the eigenvalues of J_0 go to the fixed point $\alpha^\star = s/(1-r)$ or to infinity respectively since they are given by iterations of α_0 through the function f. Finally, we mention that the allowed values of α_0 in equation (26) are purely algebraic conditions that originate from our choice that the representations of the algebra have always a lowest-weight vector.

The interesting and certainly *unexpected* connection we have just analysed between the infinite-dimensional representations of the linear Heisenberg algebra and the classification of the different types of fixed point and their stability will become more relevant in the next section, where we shall consider the quadratic case $f(J_0) = q\,J_0^2 + r\,J_0 + s$. In this case, even the finite-dimensional representations will be connected to the fixed-point analysis through the attractors of f.

It is interesting to note that in equation (23) we obtained, considering the linear case, the well known Gauss number of m as

$$\frac{N_{m-1}^2}{N_0^2} = \frac{r^m - 1}{r - 1} = [m]_r\,. \tag{28}$$

It is possible to look at the above equation the other way round and to define a general Gauss number $[m]_{\text{general}}$ for the case of arbitrary f as

$$[m]_{\text{general}} \equiv \frac{N_{m-1}^2}{N_0^2} = \frac{f^m(x) - x}{f(x) - x}. \tag{29}$$

Of course, this definition gives

$$\begin{aligned} [m]_{\text{general}} &\longrightarrow m && \text{for} \quad f(x) = x + s \\ [m]_{\text{general}} &\longrightarrow [m]_r && \text{for} \quad f(x) = r\,x + s. \end{aligned} \tag{30}$$

Finally, it is easy to see that there is a direct relation between the linear Heisenberg algebra given in equations (19)–(21) and the standard q-oscillators. In fact, defining

$$J_0 = q^{2N}\,\alpha_0 + s\ [N]_{q^2} \tag{31}$$

$$\frac{J_+}{N_0} = a^\dagger\, q^{N/2} \tag{32}$$

$$\frac{J_-}{N_0} = q^{N/2}\, a \tag{33}$$

we see that a, $a^\dagger$ and N satisfy the usual q-oscillator relations [3]

$$\begin{aligned} &a\,a^\dagger - q\,a^\dagger\,a = q^{-N} && a\,a^\dagger - q^{-1}\,a^\dagger\,a = q^N \\ &[N, a] = -a && [N, a^\dagger] = a^\dagger. \end{aligned} \tag{34}$$

Note that the Heisenberg algebra is obtained from (31)–(33) for $q \to 1$ and $\alpha_0 = 0$.

4. The nonlinear case

In this section we consider the algebra defined by equations (1)–(3) for $f(x) = t\,x^2 + r\,x + s$. In this case the algebra becomes

$$[J_0, J_+]_r = t\,J_+\,J_0^2 + s\,J_+ \tag{35}$$

$$\left[J_0, J_-\right]_{r^{-1}} = -\frac{t}{r}\,J_0^2\,J_- - \frac{s}{r}\,J_- \tag{36}$$

$$\left[J_+, J_-\right] = -t\,J_0^2 + (1-r)\,J_0 - s. \tag{37}$$

Of course, for $t = 0$ we recover the linear (or r-deformed) Heisenberg algebra given in equations (19)–(21) and for $t = 0$ and $r = 1$ the standard Heisenberg algebra.

We focus now on the analysis of equations (6) and (12)–(15), aiming to find the finite- and infinite-dimensional representations of the above quadratic Heisenberg algebra. Following an observation made at the end of the previous section we shall find the algebra representations through the analysis and the stability of the fixed points of $f(x) = t\,x^2 + r\,x + s$ and their composed functions.

One clear way to do this is to perform a graphical analysis of the function f. Let us plot $y = f(x)$ together with $y = x$. Where the lines intersect we have $x = y = f(x)$, so the intersections are precisely the fixed points. Now, for a point x_0, different from the fixed point, in order to follow its path through iterations with the function f we perform the following steps:

(1) move vertically to the graph of $f(x)$;
(2) move horizontally to the graph of $y = x$ and
(3) repeat steps (1), (2) etc (in figure 1 is shown the example of the Heisenberg algebra, where $f(J_0) = J_0 + 1$).

There are three cases to be analysed: (i) $\Delta < 0$, (ii) $\Delta = 0$ and (iii) $\Delta > 0$, for $\Delta = (r-1)^2 - 4\,t\,s$. In the first case there is no fixed point and it is easy to see by a graphical analysis that only $t > 0$ corresponds to infinite-dimensional representations ($N_m^2 \neq 0$, $\forall m$,

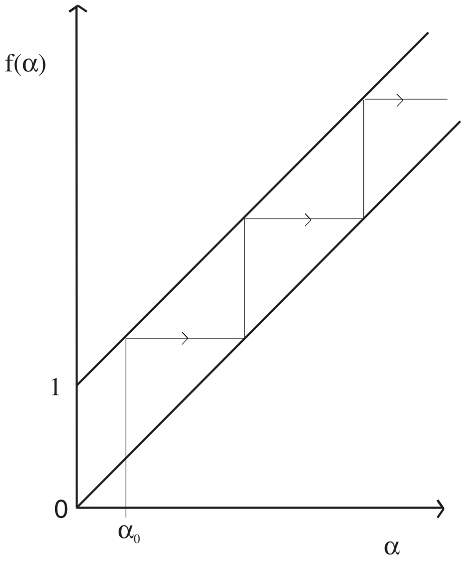

Figure 1. Iterations of α_0 for the Heisenberg algebra. The eigenvalues α_n increase by a constant factor as n increases.

$m \in Z^+$) having *lowest* weight states as desired (see figure 2(a)). Then, case (i) provides infinite-dimensional representations with lowest weight α_0 for the value of the parameters

$$t > 0: \qquad (r-1)^2 - 4ts < 0 \qquad \text{and} \qquad \alpha_0 \in \mathbb{R}. \tag{38}$$

In case (ii), $t > 0$ as well and we have one fixed point given by $\alpha^\star = (1-r)/2t$. This fixed point corresponds to a trivial one-dimensional representation of the algebra for $\alpha_0 = \alpha^\star$ since $N_0 = 0$. Besides this trivial one-dimensional representation we have for case (ii) infinite-dimensional representations with lowest weight α_0 for the set of parameters (see figure 2(b))

$$t > 0: \qquad (r-1)^2 - 4ts = 0 \qquad \text{and} \qquad \alpha_0 \in \mathbb{R} \qquad \alpha_0 \neq (1-r)/2q. \tag{39}$$

Case (iii) is less trivial. In this case it is also possible to have attractors of period $1, 2, 4, \ldots$ and even a chaotic region in the space of parameters (t, r, s, α_0). Thus, there are regions in this space associated with finite- and infinite-dimensional representations. In what follows, we analyse completely the cases of attractors of periods 1 and 2 and give an example of the chaotic behaviour of the algebra. For shortness, the analysis from now on will be done only for $t > 0$; the $t < 0$ behaviour is similar, with no conceptually significant difference.

We recall that a fixed point $\alpha^\star$, where by definition $\alpha^\star$ is the solution of the equation $\alpha^\star = f(\alpha^\star)$, is stable if $|f'(\alpha^\star)|$ is smaller than unity and is unstable if it is greater than unity. For case (iii) the fixed points are

$$\alpha^\star_\pm = \frac{1 - r \pm \sqrt{\Delta}}{2t}. \tag{40}$$

The fixed point $\alpha^\star_+$ is always unstable and computing the derivative of f at $\alpha^\star_-$ we have that $\alpha^\star_-$ is stable for a set of t, r and s such that $0 < \Delta < 4$ (we stress again that this analysis is for $t > 0$). For this set of (t, r, s) we must search for the region of α_0 that corresponds to lowest-weight states. It is easy to realize that the region $\alpha^\star_- < \alpha_0 < \alpha^\star_+$ has to be eliminated since it does not correspond to a representation with lowest-weight state; i.e., there will always exist an $n > 0$ such that $\alpha_n < \alpha_0$ if $\alpha^\star_- < \alpha_0 < \alpha^\star_+$.

For the allowed values of α_0 corresponding to infinite-dimensional representations with lowest-weight state, i.e. $-\infty < \alpha_0 < \alpha^\star_-$ and $\alpha_0 > \alpha^\star_+$, there are two types of asymptotic

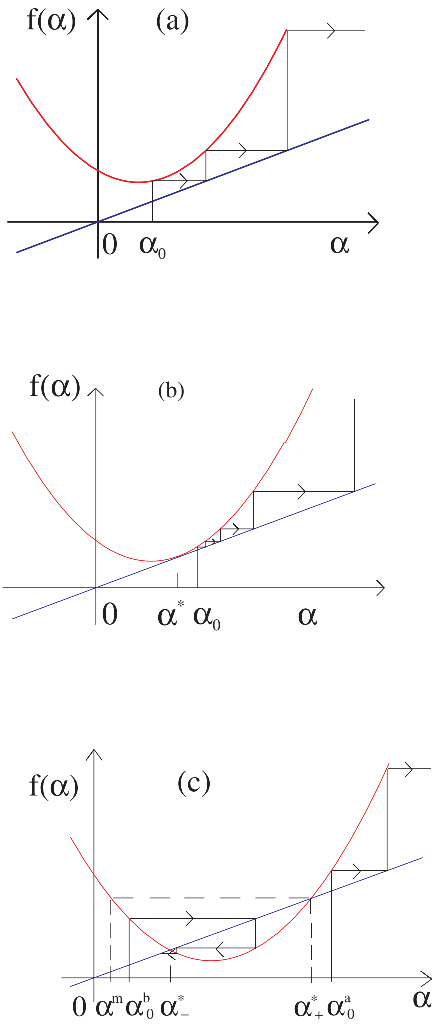

Figure 2. (*a*) Iterations of α_0 for case (i): $\Delta < 0$. As is easily seen, α_n goes to infinity as $n \to \infty$. This figure was plotted for the values $t = 1$, $r = -1.5$ and $s = 2.5$. (*b*) Iterations of α_0 for case (ii): $\Delta = 0$. Also in this case, for $\alpha \neq \alpha^\star$, α_n goes to infinity as $n \to \infty$. This figure was plotted for the values $t = 1$, $r = -2$ and $s = 9/4$. (*c*) Iterations of α_0 for case (iii): $0 < \Delta < 4$. α_0^a is a starting point belonging to the regions $\alpha_0 < \alpha^m$ or $\alpha_0 > \alpha_+^\star$, whose future iterations tend to infinity; α_0^b is a starting point belonging to the region $\alpha^m < \alpha_0 < \alpha_-^\star$, and whose future iterations tend to the fixed point $\alpha_-^\star$. This figure was plotted for the values $t = 0.8$, $r = -4$ and $s = 6$. (This figure is in colour only in the electronic version, see `www.iop.org`)

behaviour for the eigenvalues of J_0. They can go to infinity or go to the fixed point $\alpha_-^\star$. In order to identify these two regions consider the point $f(\alpha_+^\star)$. There is another point, denoted α^m, that gives $f(\alpha_+^\star)$, i.e. $f(\alpha^m) = f(\alpha_+^\star) = \alpha_+^\star$; this point is given by

$$\alpha^m = \frac{-1 - r - \sqrt{\Delta}}{2t}. \tag{41}$$

It is easy to verify that the set of (t, r, s, α_0) such that

$$0 < \Delta < 4 \qquad \text{and} \qquad \begin{cases} \text{(a)} \quad -\infty < \alpha_0 < \alpha^m \quad \text{or} \quad \alpha_+^\star < \alpha_0 < \infty \\ \text{(b)} \quad \alpha^m < \alpha_0 < \alpha_-^\star \end{cases} \tag{42}$$

corresponds to infinite-dimensional representations where the asymptotic eigenvalues of J_0 in case (a) go to infinity and in case (b) go to the asymptotic value $\alpha_-^\star$ (see figure 2(*c*)). Moreover, $\Delta > 0$ and $\alpha_0 = \alpha_-^\star$ or $\alpha_0 = \alpha_+^\star$ correspond to the trivial finite one-dimensional representation. Note that in case (b), equation (42), future iterations of α_0 (that are always bigger than α_0) will not increase monotonically. This is a specific example where a non-monotonic function f presents a non-monotonic behaviour of iterations of α_0, with a consistent vacuum $|0\rangle$.

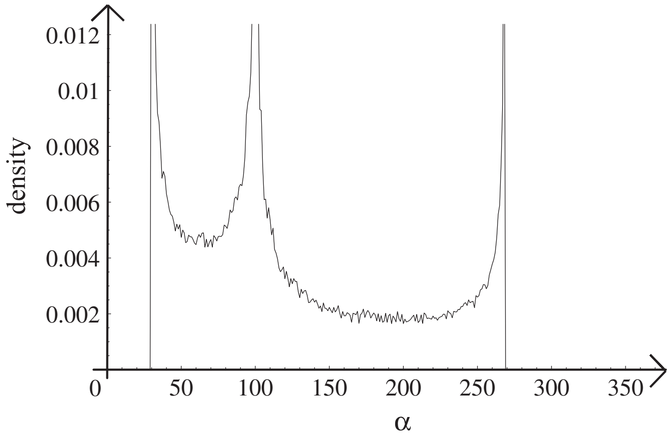

Figure 3. Histogram of the chaotic bands corresponding to the points $t = 1$, $r =$ and $s = -1.543\,591$.

The next step is to consider the set of parameters $(t,\ r,\ s,\ \alpha_0)$ such that the function $f(\alpha) = t\,\alpha^2 + r\,\alpha + s$ has an attractor of period 2. This will permit us to find infinite-dimensional representations where the asymptotic behaviour of the eigenvalues of J_0 is infinity or an attractor of period 2. Moreover, when the weight of the representation is the lowest value of the attractor there will be a set of parameters (t, r, s) corresponding to a two-dimensional representation.

In order to perform that analysis we must study the fixed points of $f^2(\beta) \equiv f(f(\beta))$, i.e. the points $\beta^\star$ satisfying $\beta^\star = f^2(\beta^\star)$ that are different from the previous one-cycle (attractors of period 1). They are

$$\beta^\star_\pm = \frac{-1 - r \pm \sqrt{\Delta_1}}{2\,t} \tag{43}$$

where $\Delta_1 = -3 - 2\,r + r^2 - 4\,t\,s$. Since the fixed points of f^2, $\beta^\star_\pm$, have the same tangent it is sufficient to analyse the stabilization region for one of them. It is simple to see that this region is given by the set (t, r, s) such that $4 < \Delta < 6$. We see that for $\Delta = 4$ the one-cycle solution loses stability and starts the stabilization region for the two-cycle solution. Then, the set of (t, r, s, α_0) such that

$$4 < \Delta < 6 \qquad \text{and} \qquad \begin{cases} \text{(c)} \quad -\infty < \alpha_0 < \alpha^m \quad \text{or} \quad \alpha^\star_+ < \alpha_0 < \infty \\ \text{(d)} \quad \alpha^m < \alpha_0 < \beta^\star_- \end{cases} \tag{44}$$

corresponds to infinite-dimensional representations where the asymptotic eigenvalues of J_0 in case (c) go to infinity and in (d) go to the lowest value of the stable two-cycle attractor with values $\beta^\star_\pm$.

In this case there is also a set of parameters, for $\Delta > 4$, corresponding to a two-dimensional representation. Note that if we take the weight of the representation as

$$\alpha_0 = \beta^\star_- = \frac{-1 - r - \sqrt{\Delta_1}}{2\,t} \tag{45}$$

we have a two-dimensional representation with matrix representation given by

$$J_0 = \begin{pmatrix} \beta^\star_- & 0 \\ 0 & \beta^\star_+ \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & 0 \\ N_0 & 0 \end{pmatrix} \qquad J_- = J_+^\dagger \tag{46}$$

where N_0 is computed for $\Delta > 4$ and α_0 given in equation (45).

Clearly, for $\Delta > 6$, we will have other cycles, of length $4, 8, \ldots, 2^k \ldots$, entering the chaotic region and displaying, in the region $(\alpha_m, \alpha_+^\star)$, exactly the same scenario the logistic map shows. To give an example of the chaotic region one chooses a point in the parameter space presenting two chaotic bands. This point corresponds to the numeric values $t = 1$, $r = 2$ and $s = -1.543\,591$ (see figure 3). Actually, there is a whole surface in the parameter space (t, r, s), in which this point is included, exhibiting these two chaotic bands. Clearly also, chaos implies infinite-dimensional representation and, for the example above, the eigenvalues of J_0 belong, mainly, to the α-region limited by the two chaotic bands shown in figure 3. The frequency of a specific eigenvalue is given by the relative height of the band at this value. If we call the lowest value of α of the two bands α^m_{chaos}, the allowed range for the lowest weight values of possible representations in this example is $\alpha_0 \in (\alpha^m, \alpha^m_{\text{chaos}})$.

In the case where $t < 0$ the whole region outside the interval $(\alpha^m, \alpha_+^\star)$ is not allowed, in contrast to the case $t > 0$. The lowest fixed point is always unstable, also in contrast to the case of positive values of t, where the highest fixed point was always unstable, but the general sequence of attractors and chaotic regions is exactly the same, as is well known. A study of a particular case of $t < 0$, the logistic case, was made in [6, 7].

5. Final comments

In this paper we have presented the first steps towards the complete analysis of the algebra described by the relations in equations (1)–(3). This algebra can be rewritten for the polynomial $f(J_0) = \sum_{i=0}^{n} a_i J_0^i$ as

$$[J_0, J_+]_{a_1} = a_0 J_+ + \sum_{i=2}^{n} a_i J_+ J_0^i \tag{47}$$

$$\left[J_0, J_-\right]_{a_1^{-1}} = -\frac{a_0}{a_1} J_- - \sum_{i=2}^{n} \frac{a_i}{a_1} J_0^i J_- \tag{48}$$

$$\left[J_+, J_-\right] = -\sum_{i=2}^{n} a_i J_0^i + (1 - a_1)\, J_0 - a_0. \tag{49}$$

The linear case, $f(J_0) = a_0 + a_1 J_0$, corresponds to the Heisenberg algebra for $a_1 = 1$ and to the a_1^2-deformed Heisenberg algebra otherwise. The representation theory has been shown to be directly related to the stability analysis of the fixed point of the function f and their composed functions.

The linear and quadratic cases have been analysed in detail. The finite-dimensional representations correspond to lowest weight being the lowest value of the attractors of period $1, 2, 4, \ldots$. Moreover, associated with each attractor there is a parameter region providing an infinite-dimensional representation. We expect that this relation between representations and stability analysis of the fixed points of f and their composed functions will be the same for any analytical function f. In fact, in higher-order polynomials there will be the possibility to have, simultaneously, more than one attractor, each one with its own basin of attraction in the parameter space. In spite of this, inside one particular basin of attraction the scenario is the same as analysed here in the nonlinear case.

It is interesting to mention that there are parameter regions corresponding to certain representations that cannot be smoothly deformed to a representation of Heisenberg algebra. An obvious example is the so-called logistic algebra where $f(J_0) = r\, J_0(1 - J_0)$ is chosen as the logistic map for J_0. It is clear that this algebra cannot be deformed to the Heisenberg algebra even if it is a generalization of it in the sense discussed in this paper.

Last, but not least, we have the feeling that the approach we have presented in this paper is, in a certain sense, universal. In this approach we construct the nonlinear generalization of a given undeformed algebra and its representation theory is directly related to the classification of the fixed points—and their stability—of a function f (and the composed functions) that generates the algebra.

In fact, it is possible to construct another iterative algebra as

$$J_0\, J_- = J_-\, f(J_0) \tag{50}$$

$$J_+\, J_0 = f(J_0)\, J_+ \tag{51}$$

$$\left[J_+, J_-\right] = J_0(J_0 + 1) - f(J_0)(f(J_0) + 1) \tag{52}$$

with Casimir

$$C = J_+\, J_- + f(J_0)(f(J_0) + 1) = J_-\, J_+ + J_0(J_0 + 1) \tag{53}$$

where $J_- = J_+^\dagger$, $J_0^\dagger = J_0$ and $f(J_0)$ is an analytical function in J_0. Note that if $f(J_0)$ is the simplest linear functional $f(J_0) = J_0 - 1$ we obtain the relations and the Casimir of the $su(2)$ algebra. It is tempting to investigate, as we did in this paper for the iterative algebra in equations (1)–(3), the above algebra for more complicated functionals $f(J_0)$.

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